

Evaluations For $\varsigma(2)$, $\varsigma(4)$, \dots , $\varsigma(2k)$ Based On The WZ Method

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Abstract

Based on the framework of the WZ theory, a new evaluation for $\varsigma(2) = \frac{\pi^2}{6}$ and $\varsigma(4) = \frac{\pi^4}{90}$ was given respectively, finally, a recurrence formula for $\varsigma(2k)$, which is equivalent to the classical formula $B_{2k}(\frac{1}{2}) = (2^{-2k+1} - 1)B_{2k}$, was given.

1 Introduction, Lemmas and Main Results.

We know that there are many evaluations (or proofs) for $\varsigma(2) = \frac{\pi^2}{6}$ since the first evaluation belong to Euler, e.g. see [1]-[5] and the related references therein. We also know that there are two recurrence formulas for $\varsigma(2k)$ in [5]

$$\begin{aligned}\varsigma(2n) &= (-1)^{n-1} \frac{(2\pi)^{2n-1}}{(2n)! (2^{2n-1} - 1)} \left[\frac{\pi}{2(2n+1)} + \sum_{j=1}^{n-1} (-1)^j \binom{2n}{2j-1} \frac{(2j-1)! (2^{2j-1} - 1)}{(2\pi)^{2j-1}} \varsigma(2j) \right], \\ \varsigma(2n) &= (-1)^{n-1} \frac{(2\pi)^{2n-1}}{(2n-1)! (2^{2n-1} - 1)} \left[\frac{\pi}{4n} + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} \frac{(2j-1)! (2^{2j-1} - 1)}{(2\pi)^{2j-1}} \varsigma(2j) \right].\end{aligned}$$

In this paper, I will give a new evaluation for $\varsigma(2) = \frac{\pi^2}{6}$ based on the framework of WZ theory (see [6]-[8]), repeating the process of evaluation can also be applied to evaluating $\varsigma(4)$, $\varsigma(6)$, \dots . Finally, through the same process repeatedly, I obtained a recurrence formula for $\varsigma(2k)$ which is similar to, but different from the recurrence formulas for $\varsigma(2k)$ mentioned above. As $\varsigma(2k) = \frac{2^{2k-1}(-1)^{k-1}B_{2k}\pi^{2k}}{(2k)!}$ (belong to Euler too), and B_k (where $k \in N_0 = N \cup \{0\}$) is called k -th Bernoulli number, thus by the recurrence formula for $\varsigma(2k)$ in the following theorem, we can obtain a formula for Bernoulli polynomial B_{2k} : $B_{2k}(\frac{1}{2}) = (2^{-2k+1} - 1)B_{2k}$, where $B_n(x)$ is the Bernoulli polynomial of order n , in fact, there are equivalent. The following theorem is the main result in this paper.

Theorem. Given $\varsigma(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ (where $Re(s) > 1$), then we have $\varsigma(2) = \frac{\pi^2}{6}$, $\varsigma(4) = \frac{\pi^4}{90}$, more generally with the convention $\sum_{k=1}^0 a(k) = 0$, for $\varsigma(2l)$ (where

$l \in N$) the following recurrence formula hold

$$\varsigma(2l) = \left(\frac{2^{2l-1}}{1-2^{2l}} \right) \left\{ \left[\frac{(-1)^{l+1}}{4l} + \frac{(-1)^l}{2} \right] \frac{\pi^{2l}}{\Gamma(2l)} + \sum_{j=1}^{l-1} \frac{(-1)^{l-j} \pi^{2(l-j)}}{\Gamma(2(l-j)+1)} \varsigma(2j) \right\}$$

where $\Gamma(z)$ is gamma function.

To prove the theorem, we need the following lemmas.

Lemma 1. If given a continuous-discrete WZ pair $(F(x, k), G(x, k))$, that is $(F(x, k), G(x, k))$ satisfy the following so called continuous-discrete WZ equation

$$\frac{\partial F(x, k)}{\partial x} = G(x, k+1) - G(x, k) \quad (1)$$

then for all $m, n \in N_0$, for all $h, x \in R$, we have

$$\sum_{k=m}^n F(x, k) - \sum_{k=m}^n F(h, k) = \int_h^x G(t, n+1) dt - \int_h^x G(t, m) dt.$$

Lemma 2. If for $a, x \in R$, $a < x$, $f(x)$ is integrable on the interval (a, x) , then we have

$$\int_a^x \left(\int_a^{t_1} \cdots \left(\int_a^{t_{k-1}} f(t_k) dt_k \right) \cdots dt_2 \right) dt_1 = \frac{1}{\Gamma(k)} \int_a^x (x-t)^{k-1} f(t) dt.$$

Lemma 3. For all $n \in N$, we have

$$\sum_{k=1}^n \cos kx = -\frac{1}{2} + \frac{1}{2} \frac{\sin[(2n+1)x/2]}{\sin(x/2)}.$$

Lemma 4. For all $n \in N_0$, we have

$$\int_0^\pi \frac{\sin[(2n+1)x/2]}{\sin(x/2)} dx = \pi.$$

Lemma 5. For all $s \geq 1$, we have

$$\lim_{n \rightarrow +\infty} \int_0^\pi \frac{x^s \sin[(2n+1)x/2]}{\sin(x/2)} dx = 0.$$

As the proof of **Lemma 1** is easy, we omit the details of proof here. **Lemma 2** can be seen in [9]-[10], **Lemma 3** and **Lemma 4** can be seen in [11]. The proof of **Lemma 5** will be given below.

2 Proof of Lemma 5.

Let $t = \frac{x}{2}$, then we have

$$\int_0^\pi \frac{x^s \sin[(2n+1)x/2]}{\sin(x/2)} dx = 2^{s+1} \int_0^{\frac{\pi}{2}} \frac{t^s \sin[(2n+1)t]}{\sin(t)} dt.$$

Let $f(t) = \begin{cases} t^s \csc t & 0 < t < \frac{\pi}{2} \\ 0 & t = 0 \end{cases}$, it is an easy exercise of calculus that when $s \geq 1$, $f(t)$ is differentiable and monotone (increasing) on $[0, \frac{\pi}{2}]$, then by **The Second Mean Value Theorem For Integrals**, we know that there exist ξ on $[0, \frac{\pi}{2}]$ such that

$$\begin{aligned} & \left| \int_0^{\frac{\pi}{2}} \frac{t^s}{\sin t} \sin((2n+1)t) dt \right| \\ &= \left| \int_0^{\frac{\pi}{2}} f(t) \sin((2n+1)t) dt \right| \\ &= \left| f(0+0) \int_0^\xi \sin((2n+1)t) dt + f\left(\frac{\pi}{2}-0\right) \int_\xi^{\frac{\pi}{2}} \sin((2n+1)t) dt \right| \\ &= \left(\frac{\pi}{2} \right)^s \left| \frac{1}{2n+1} (-\cos((2n+1)t)) \Big|_{\frac{\pi}{2}}^\xi \right| \\ &\leq \left(\frac{\pi}{2} \right)^s \frac{2}{2n+1}. \end{aligned}$$

By the result above, we can conclude that

$$\lim_{n \rightarrow +\infty} \int_0^\pi \frac{x^s \sin\left(\frac{(2n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx = \lim_{n \rightarrow +\infty} 2^{s+1} \int_0^{\frac{\pi}{2}} \frac{t^s \sin((2n+1)t)}{\sin t} dt = 0,$$

the proof of **Lemma 5** was completed.

Remarks: 1. It is worth mentioning that we can prove **Lemma 5** by Riemann-Lebesgue lemma directly as follows. Let $f(t) = \begin{cases} \left(\frac{t}{2}\right)^s \csc\left(\frac{t}{2}\right) & 0 < t < \pi \\ 0 & t = 0 \end{cases}$, it is an easy exercise of calculus that $f(t)$ is continuous on $[0, \pi]$, of course, $f(t)$ is Riemann integrable on $[0, \pi]$, then by Riemann-Lebesgue lemma, we have

$$\lim_{n \rightarrow +\infty} \int_0^\pi f(x) \sin\left(\frac{(2n+1)x}{2}\right) dx = 0,$$

because

$$\int_0^\pi f(x) \sin\left(\frac{(2n+1)x}{2}\right) dx = \int_0^\pi \frac{\left(\frac{x}{2}\right)^s \sin\left(\frac{(2n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx,$$

finally, we have

$$\lim_{n \rightarrow +\infty} \int_0^\pi \frac{x^s \sin\left(\frac{(2n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx = 0.$$

2. It is also worth mentioning that when $s \geq 2$, we can prove **Lemma 5** by using integration by parts, but when $1 \leq s < 2$, the method can't be used.

3 Proof of The Theorem.

(A) Proof of $\varsigma(2) = \frac{\pi^2}{6}$. Setting $F_1(x, k) = \frac{\cos(kx)}{k^2}$, $G_1(x, k) = \sum_{j=1}^{k-1} \frac{-\sin(jx)}{j}$, then it is easy to verify that $(F_1(x, k), G_1(x, k))$ is a continuous-discrete WZ pair, that is, they satisfy the equation (1). Now let $h = 0$, $m = 1$, with the convention $\sum_{k=1}^0 a(k) = 0$, by using **Lemma 1** we get

$$\sum_{k=1}^n \frac{\cos(kx)}{k^2} - \sum_{k=1}^n \frac{1}{k^2} = \int_0^x G_1(t, n+1) dt \quad (2)$$

To evaluate $G_1(x, n+1) = \sum_{j=1}^n \frac{-\sin(jx)}{j}$, we also use **Lemma 1**. Now set

$$F_2(x, k) = \frac{-\sin(kx)}{k}, \quad G_2(x, k) = \sum_{j=1}^{k-1} -\cos(jx),$$

then it is easy to verify that $(F_2(x, k), G_2(x, k))$ is a continuous-discrete WZ pair, and for all $k \in N$, the following result hold $F_2(0, k) = 0$. With the convention $\sum_{k=1}^0 a(k) = 0$, by using **Lemma 1** and **Lemma 3**, we obtain

$$\sum_{k=1}^n \frac{-\sin(kx)}{k} = \int_0^x G_2(t, n) dt = \int_0^x \left\{ \frac{1}{2} - \frac{\sin[(2n+1)t/2]}{2 \sin(t/2)} \right\} dt \quad (3)$$

By using (2), (3) and **Lemma 2**, we obtain

$$\begin{aligned} \sum_{k=1}^n \frac{\cos(kx)}{k^2} - \sum_{k=1}^n \frac{1}{k^2} &= \int_0^x G_1(t, n+1) dt \\ &= \int_0^x \left\{ \int_0^{t_1} \left[\frac{1}{2} - \frac{\sin[(2n+1)t_2/2]}{2 \sin(t_2/2)} \right] dt_2 \right\} dt_1 \\ &= \frac{1}{2} \int_0^x (x-t) dt - \frac{x}{2} \int_0^x \frac{\sin[(2n+1)t/2]}{\sin(t/2)} dt \\ &\quad + \frac{1}{2} \int_0^x \frac{t \sin[(2n+1)t/2]}{\sin(t/2)} dt \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

Recalling $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k^s} - \varsigma(s) = \left(-2 + \frac{1}{2^{s-1}}\right) \varsigma(s)$, let $x = \pi$ at first, and then let $n \rightarrow +\infty$, we conclude that

$$\lim_{n \rightarrow +\infty} \left[\sum_{k=1}^n \frac{\cos(k\pi)}{k^2} - \sum_{k=1}^n \frac{1}{k^2} \right] = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^2} - \varsigma(2) = -\frac{3}{2} \varsigma(2).$$

After some computations, we obtain

$$I_1(\pi) = \frac{1}{2} \int_0^\pi (\pi - t) dt = \frac{\pi^2}{4}.$$

By **Lemma 4**, we obtain

$$I_2(\pi) = -\frac{\pi}{2} \int_0^\pi \frac{\sin[(2n+1)t/2]}{\sin(t/2)} dt = -\frac{\pi^2}{2}.$$

Now by **Lemma 5**, we obtain

$$\lim_{n \rightarrow +\infty} I_3(\pi) = \lim_{n \rightarrow +\infty} \frac{1}{2} \int_0^\pi \frac{t \sin[(2n+1)t/2]}{\sin(t/2)} dt = 0.$$

Finally we conclude that $-\frac{3}{2} \varsigma(2) = \frac{\pi^2}{4} - \frac{\pi^2}{2} = -\frac{\pi^2}{4}$, that is $\varsigma(2) = \frac{\pi^2}{6}$.

(B) Proof of $\varsigma(4) = \frac{\pi^4}{90}$. Setting

$$\begin{aligned} F_1(x, k) &= \frac{\cos(kx)}{k^4}, & G_1(x, k) &= \sum_{j=1}^{k-1} \frac{-\sin(jx)}{j^3}, \\ F_2(x, k) &= \frac{-\sin(kx)}{k^3}, & G_2(x, k) &= \sum_{j=1}^{k-1} \frac{-\cos(jx)}{j^2}, \\ F_3(x, k) &= \frac{-\cos(kx)}{k^2}, & G_3(x, k) &= \sum_{j=1}^{k-1} \frac{\sin(jx)}{j}, \\ F_4(x, k) &= \frac{\sin(kx)}{k}, & G_4(x, k) &= \sum_{j=1}^{k-1} \cos(jx), \end{aligned}$$

It is easy to verify that for $j = 1, 2, 3, 4$, $(F_j(x, k), G_j(x, k))$ satisfy equation (1).

Setting $H_n^{(l)} = \sum_{k=1}^n \frac{1}{k^l}$, completely analogous to the proof of $\varsigma(2) = \frac{\pi^2}{6}$ (some details are omitted here.), we get

$$\begin{aligned}
\sum_{k=1}^n \frac{\cos(kx)}{k^4} - H_n^{(4)} &= \frac{1}{\Gamma(4)} \left\{ \left[\int_0^x -\frac{1}{2}(x-t)^3 dt + \frac{x^3}{2} \int_0^x \frac{\sin[(2n+1)t/2]}{\sin(t/2)} dt \right] \right\} \\
&+ \frac{1}{\Gamma(4)} \left\{ \sum_{k=1}^2 \binom{3}{k} \frac{x^k}{2} \int_0^x (-t)^{3-k} \frac{\sin[(2n+1)t/2]}{\sin(t/2)} dt \right\} \\
&- \frac{1}{\Gamma(4)} \int_0^x (x-t) H_n^{(2)} dt \\
&= I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

Setting $x = \pi$, by **Lemma 4** we obtain $I_1(\pi) = \frac{\pi^4}{16}$, and by **Lemma 5** we obtain $\lim_{n \rightarrow +\infty} I_2(\pi) = 0$, recall $\lim_{n \rightarrow +\infty} H_n^{(2)} = \zeta(2) = \frac{\pi^2}{6}$, we obtain $\lim_{n \rightarrow +\infty} I_3(\pi) = -\frac{\pi^4}{12}$. Recall $\lim_{n \rightarrow +\infty} H_n^{(4)} = \zeta(4)$, we get

$$\lim_{n \rightarrow +\infty} \left[\sum_{k=1}^n \frac{\cos(k\pi)}{k^4} - H_n^{(4)} \right] = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^4} - \zeta(4) = \left(-2 + \frac{1}{2^3} \right) \zeta(4).$$

Finally we obtain $\zeta(4) = \frac{\pi^4}{90}$.

$$\textbf{(C) Proof of } \zeta(2l) = \left(\frac{2^{2l-1}}{1-2^{2l}} \right) \left\{ \left[\frac{(-1)^{l+1}}{4l} + \frac{(-1)^l}{2} \right] \frac{\pi^{2l}}{\Gamma(2l)} + \sum_{j=1}^{l-1} \frac{(-1)^{l-j} \pi^{2(l-j)}}{\Gamma(2(l-j)+1)} \zeta(2j) \right\}.$$

The result can be proved in the above framework of proving $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$, some details are omitted here. For convenience, setting

$$H_n^{(l)}(x) = \sum_{k=1}^n \frac{\cos(kx)}{k^l}, \quad H_n^{(l)}(0) = H_n^{(l)}, \quad H_n^{(l)}(\pi) = \sum_{k=1}^n \frac{(-1)^k}{k^l},$$

$$I_j(f)(x) = \frac{1}{\Gamma(j)} \int_0^x (x-t)^{j-1} f(t) dt,$$

with the convention that $I_0(f)(x) = f(x)$, where $j \in N_0$. Now it is easy to verify that for all $j \in N_0$, I_j own the following properties

$$I_j(f+g)(x) = I_j(f)(x) + I_j(g)(x), \quad I_j(cf)(x) = cI_j(f)(x),$$

where c is a constant having nothing to do with t , the variable of integral. Also setting

$$f(t) = -\frac{1}{2} + \frac{1}{2} \frac{\sin[(2n+1)t/2]}{\sin(t/2)},$$

we obtain

$$H_n^{(2l)}(x) = (-1)^l I_{2l}(f)(x) + \sum_{j=1}^l (-1)^{l-j} I_{2(l-j)}(H_n^{(2j)})(x).$$

Let $x = \pi$, then we get the following result

$$\begin{aligned}
\sum_{k=1}^n \frac{(-1)^k}{k^{2l}} - H_n^{(2l)} &= (-1)^l I_{2l} \left(-\frac{1}{2} \right) (\pi) + (-1)^l I_{2l} \left(\frac{\sin [(2n+1)t/2]}{2 \sin(t/2)} \right) (\pi) \\
&\quad + \sum_{j=1}^{l-1} (-1)^{l-j} I_{2(l-j)} (H_n^{(2j)}) (\pi) \\
&= \text{I} + \text{II} + \text{III}.
\end{aligned}$$

Next, let us consider I, II and III respectively

$$\begin{aligned}
\text{I} &= (-1)^l \left(-\frac{1}{2} \right) \frac{1}{\Gamma(2l)} \int_0^\pi (\pi - t)^{2l-1} dt = \frac{(-1)^{l+1}}{4l} \frac{\pi^{2l}}{\Gamma(2l)} \\
\text{II} &= (-1)^l \frac{1}{2\Gamma(2l)} \int_0^\pi (\pi - t)^{2l-1} \frac{\sin [(2n+1)t/2]}{\sin(t/2)} dt \\
&= (-1)^l \frac{1}{2\Gamma(2l)} \int_0^\pi \pi^{2l-1} \frac{\sin [(2n+1)t/2]}{\sin(t/2)} dt \\
&\quad + (-1)^l \frac{1}{2\Gamma(2l)} \int_0^\pi \sum_{k=1}^{2l-2} \binom{2l-1}{k} \pi^k (-t)^{2l-1-k} \frac{\sin [(2n+1)t/2]}{\sin(t/2)} dt \\
&= \text{II}_1 + \text{II}_2.
\end{aligned}$$

By Lemma 4, we obtain

$$\text{II}_1 = (-1)^l \frac{\pi^{2l}}{2\Gamma(l)},$$

and by Lemma 5 we conclude that $\lim_{n \rightarrow +\infty} \text{II}_2 = 0$. By using the results above, we obtain

$$\lim_{n \rightarrow +\infty} \text{II} = (-1)^l \frac{\pi^{2l}}{2\Gamma(l)}.$$

After some computations, we obtain

$$\begin{aligned}
\text{III} &= \sum_{j=1}^{l-1} (-1)^{l-j} H_n^{(2j)} \frac{1}{\Gamma(2(l-j))} \int_0^\pi (\pi - t)^{2(l-j)-1} dt \\
&= \sum_{j=1}^{l-1} (-1)^{l-j} H_n^{(2j)} \frac{\pi^{2(l-j)}}{\Gamma(2(l-j)+1)}.
\end{aligned}$$

Recalling $\lim_{n \rightarrow +\infty} H_n^{(2j)} = \zeta(2j)$, we conclude that

$$\lim_{n \rightarrow +\infty} \text{III} = \sum_{j=1}^{l-1} \frac{(-1)^{l-j} \pi^{2(l-j)}}{\Gamma(2(l-j)+1)} \varsigma(2j).$$

It is easy to verify that

$$\lim_{n \rightarrow +\infty} \left(\sum_{k=1}^n \frac{(-1)^k}{k^{2l}} - \sum_{k=1}^n \frac{1}{k^{2l}} \right) = \left(-2 + \frac{1}{2^{2l-1}} \right) \varsigma(2l).$$

So finally with the convention $\sum_{k=1}^0 a(k) = 0$, we obtain the following recurrence formula for $\varsigma(2k)$

$$\varsigma(2l) = \left(\frac{2^{2l-1}}{1-2^{2l}} \right) \left\{ \left[\frac{(-1)^{l+1}}{4l} + \frac{(-1)^l}{2} \right] \frac{\pi^{2l}}{\Gamma(2l)} + \sum_{j=1}^{l-1} \frac{(-1)^{l-j} \pi^{2(l-j)}}{\Gamma(2(l-j)+1)} \varsigma(2j) \right\}.$$

The proof of the theorem is completed.

Remarks: 1. We can set

$$F_1(x, k) = \frac{e^{ikx}}{k^2}, \quad G_1(x, k) = \sum_{j=1}^{k-1} \frac{ie^{ijx}}{j},$$

$$F_2(x, k) = \frac{ie^{ikx}}{k}, \quad G_2(x, k) = \sum_{j=1}^{k-1} -e^{ijx}$$

where $i = \sqrt{-1}$, it is easy to verify that $(F_j(x, k), G_j(x, k))$ (where $j = 1, 2$) is a continuous-discrete WZ pair. Then through the same process above, we can also obtained $\varsigma(2) = \frac{\pi^2}{6}$, the details are omitted here. Of course, we can do in the same way for $\varsigma(4)$, $\varsigma(6)$, \dots , and for the general case $\varsigma(2k)$ respectively. 2. It is also worth mentioning that the ideas in the proof above can be used to solve other similar problems of summation of infinite series, and I will give the details in another paper.

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